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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. SIXTH SEMESTER EXAMINATION, MAY 2018
THIRD YEAR [BATCH 2015-18]
MATHEMATICS (Honours)
Paper : VII

Date : 04/05/2018
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book <u>for each Group</u>]

<u>Group - A</u>

(*Answer **<u>any three</u>** questions*) [3×10]

1. a) Examine the convergence of $\int_0^1 x^{m-1}(1-x)^{n-1}\log x\,dx$. [5]

   b) Show that $\iint_E \frac{\sqrt{a^2b^2-b^2x^2-a^2y^2}}{\sqrt{a^2b^2+b^2x^2+a^2y^2}}dxdy=\frac{\pi}{4}\left(\frac{\pi}{2}-1\right)ab$, where E is the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. [5]

2. a) Prove that $\int_0^\infty \frac{\sin mx}{x^n}dx\ (m,\ n>0)$ is convergent if , 0<n<2 and absolutely convergent if 1< n<2. [5]

   b) Find the Fourier series expansion for e^{ax} on $(-\pi,\pi)$. [5]

3. a) Show that $\int_0^1 x^{m-1}(\log x)^{n-1}dx$ is convergent if m > 0, n > 0. [4]

   b) Show that the series $2\left\{\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+...\right\}$ represents $(\pi-x)$ on $(0,2\pi)$. [3]

   c) Change the order of the integration for $\int_{1/3}^{2/3}dx\int_{x^2}^{\sqrt{x}}f(x,y)dy$. [3]

4. a) Consider the periodic function $f:[0,2\pi]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1 & \text{if} \quad 0\le x<\pi\\ 0 & \text{if} \quad \pi\le x<2\pi\\ 1 & \text{if} \quad x=2\pi\end{cases}$$

   Find the Fourier series of f. [5]

   b) Evaluate $\iint\sqrt{4a^2-x^2-y^2}\,dxdy$, the integral being taken over upper half of the circle $x^2+y^2-2ax=0$. [5]

5. a) Show that the Fourier Series of x cos x on $[-\pi,\pi]$ is $x\cos x=-\frac{1}{2}\sin x+2\sum_{n=2}^{\infty}\frac{(-1)^n n\sin nx}{n^2-1}$. [5]

   b) Evaluate : $\int_0^\infty e^{-t^2}\cos(xt)dt$. [5]

Group - B

(*Answer **any four** questions*) [4×5]

6. a) Suppose a, b are two positive integers such that gcd (a, b) = 1. Prove that $\gcd(a+b, ab) = 1$. [3]

   b) Prove that, for any positive integer n, $\phi(3n) = 3\phi(n)$ if and only if n is divisible by 3, where ϕ is Euler phi-function. [2]

7. Find the least positive integer which leaves the remainders 3, 6 and 4 when divided by 5, 7 and 11 respectively. [5]

8. a) If p and $p^2 + 8$ both are prime numbers, prove that p = 3. [3]

   b) If p is a prime number greater than 3, then prove that $p^2 - 1$ is divisible by 24. [2]

9. State and prove Wilson's theorem. [5]

10. Define Euler's phi-function ϕ. Show that for any positive integer n, $n = \sum_{d|n} \phi(d)$ where the summation runs over all positive divisors of n. [1+4]

11. a) Show that there are no positive integer n such that $\sigma(n) = 10$. [2]

    b) Find the smallest positive integer with 18 positive divisors. [3]

Group - C

(*Answer **any three** questions*) [3×10]

12. a) State the axiomatic definition of probability. [2]

    b) Show that the conditional probabilities satisfy the three axioms of probability. [4]

    c) There are three coins, identical in appearance, one of which is unbiased and other two are biased with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively for a head. One coin is taken at random and tossed twice. If head appears both times, what is the probability that the unbiased coin is chosen? [4]

13. a) A man takes a step forward with probability 0·7 and backward with probability 0·3. Find the probability that at the end of 13 steps he is 3 steps away from the starting point. [5]

    b) Find the value of the constant K so that the function f_x given by

    $$f_x(x) = \begin{cases} Kx(2-x) & , \quad 0 < x < 2 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

    is a probability density function. Construct the distribution function and compute $P(X > 1)$. [5]

14. a) If X and Y are independent variates, both uniformly distributed over (0,1), find the distributions of X + Y and X – Y. [5]

    b) X be a normal (m, σ) variate, then prove that $\mu_{2K+2} = \sigma^2\mu_{2K} + \sigma^3 \frac{d\mu_{2K}}{d\sigma}$, μ_K is the K-th central moment. [5]

15. a) Let X, Y be two correlated random variables and U, V be two random variables obtained from the random variables X, Y by the rotation of coordinate axes through an angle α i.e

$U = X\cos\alpha + Y\sin\alpha$, $V = -X\sin\alpha + Y\cos\alpha$ then show that U and V will be uncorrelated, if $\alpha = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_x\sigma_y\rho(X,Y)}{\sigma_x^2 - \sigma_y^2}\right)$. [4]

b) Show that the acute angle θ between the least square regression lines is given by $\tan\theta = \frac{1-\rho^2}{|\rho|}\cdot\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$. Discuss the cases when $\rho = 0$ & $\rho = \pm 1$. [3]

c) Let $\{X_n\}_n$ be a sequence of random variables such that the mean m_n and variance σ_n^2 of X_n for each n exists. If $\sigma_n \to 0$ as $n \to \infty$, then show that $X_n - m_n \xrightarrow{\text{in P}} 0$ as $n \to \infty$. [3]

16. a) A random variable X has p.d.f. $12x^2(1-x)$, $0 < x < 1$. Compute $P(|X - m| \geq 2\sigma)$ and compare it with the limit given by Tchebycheff's Inequality. [5]

b) Use Tchebycheff's inequality to show that for $n \geq 36$, the probability that in n throws of a fair die the number of sixes lies between $\frac{n}{6} - \sqrt{n}$ and $\frac{n}{6} + \sqrt{n}$ is at least $\frac{31}{36}$. [5]

Group - D

(*Answer **<u>any two</u>** questions*) [2×10]

17. a) Show that the function $f(z) = |z|$ is nowhere differentiable in $\mathbb{C}$ but continuous everywhere. [4]

b) Let $f(z)$ be a real function of a complex variable z. Then show that either f has the derivative zero or the derivative does not exist. [3]

c) If $p(x,y)$ and $q(x,y)$ are harmonic functions in a domain $\Omega \subset \mathbb{C}$, prove that $c_1p + c_2q$ are harmonic in Ω where c_1, c_2 are constant. [3]

18. a) Let $f(z) = u(x,y) + i\,v(x,y)$ be an analytic function (where z = x + iy). Find f(z) when $u - v = (x-y)(x^2 + 4xy + y^2)$. [4]

b) Find the image point on the Riemann sphere S for the point 4 + 3i in the complex plane $\mathbb{C}$. [3]

c) "1 is not the maximum value of $|\sin z|$, when z is a complex variable" —justify the statement. [3]

19. a) Let $f_n \to f$ uniformly on $S(\subseteq \mathbb{C})$. If each f_n is continuous at a point c of S then show that the limit function f is also continuous at c. [4]

b) Let $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z_0 = x_0 + iy_0$. Prove that f is differentiable in the sense of real variables as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Also show that the determinant of the Jacobian matrix of f at (x_0, y_0) is $|f'(z_0)|^2$. [4+2]

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